A Deterministic Algorithm for Solving Imprecise Decision Problems

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Motivation

Problems with classical decision analysis tools:

- ▷ Numerically precise data is required.
- ▷ Data must be consistent.

Modelling Impreciseness

- Dempster-Shafer theory [Shafer, 76]:
  - Focus on representation—not decision analysis.
  - Unnecessarily strong with respect to interval representation.
- Fuzzy set theory, e.g. [Lai & Hwang, 94].
- Epistemic reliability [Gärdenfors & Sahlin, 83]:
  - Only imprecise probabilities—not utilities.
  - Only global beliefs.

Outline

- Representation
- Evaluation
- Dealing with Inconsistency
- Interval and Point Values
- Conclusions

Decision Situations

- ▷ A decision situation D consists of a set of n alternatives

$$\{\{c_{ij}\}_{j=1,\ldots,m_i}\}_{i=1,\ldots,n}$$

- ▷ An alternative is represented by a set C_i of m_i consequences.

- ▷ Each consequence c_{ij} has a probability p_{ij} of occurring, and has utility u_{ij} for the decision maker if it occurs.

Statements

▷ Exact statements:

"The probability of c_{21} is 0.17." ($p_{21} = 0.17$)

▷ Qualitative statements:

"Consequence c_{11} is very probable." ($p_{11} \in [a, b]$)

▷ Comparative statements:

"Consequence c_{12} is at least as desirable as c_{11}."
($u_{12} \geq u_{11}$)

Probability and Utility Bases

- ▷ The set of constraints involving probability variables, together with $\sum_{j=1}^{m_i} p_{ij} = 1$ for each set of consequences, is the probability base $\mathcal{P}$.
- ▷ The utility base $\mathcal{V}$ consists of all constraints involving utility variables.

Epistemologically Possible Distributions

- ▷ The solution set $E_{\mathcal{P}}$ of $\mathcal{P}$ is the set of epistemologically possible probability distributions.

- ▷ The solution set $E_{\mathcal{V}}$ of $\mathcal{V}$ is the set of epistemologically possible utility distributions.

These are the distributions in which the decision maker has a positive belief.

No variation in belief intensity!

Unit Cubes

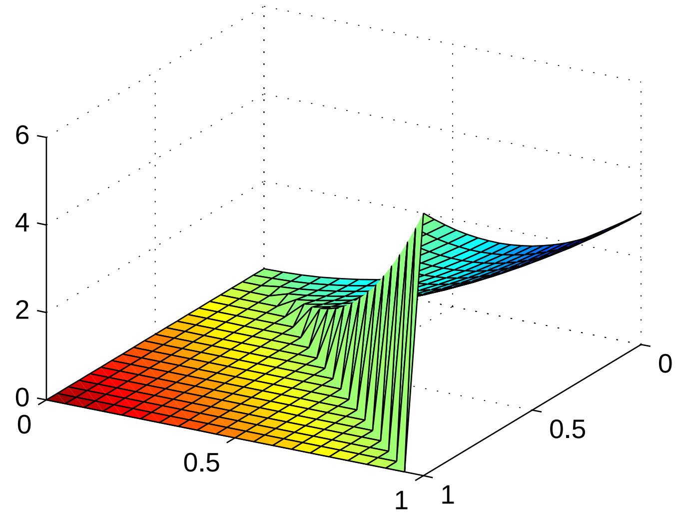
- A unit cube for a consequence c_{ij} is the interval $[0, 1]$, denoted $B = (b_{ij})$.
- A unit cube for an alternative $\{c_{ij}\}_{j=1,\ldots,m_i}$ is the space $[0, 1]^{m_i}$, denoted $B = (b_{i1}, \ldots, b_{im_i})$.
- A unit cube for a decision situation D, given unit cubes B_i for the alternatives, is the space $B_1 \times \ldots \times B_n$, denoted $B = (B_1, \ldots, B_n)$ (alt. $B = (b_1, \ldots, b_k)$).

Global Belief Distributions

Given a unit cube $B = (b_1, \ldots, b_k)$, a global belief distribution over B is a positive distribution g such that

$$\int_B g(x)dV_B(x) = 1$$

$$g(u_{11}, u_{12}) = \begin{cases} 3(u_{11}^2 + u_{12}^2) & \text{if } u_{12} \geq u_{11} \\ 0 & \text{otherwise} \end{cases}$$



Global Belief Distributions (cont.)

Global belief distributions generalizes the concept of probability and utility bases.

Given a global belief distribution g, $\mathsf{supp}\ g$ is the epistemologically possible distributions ($E_{\mathcal{P}}$ or $E_{\mathcal{V}}$).

Local Belief Distributions

Seldom, a decision maker can specify global belief distributions.

Given a unit cube $B = (b_1, \ldots, b_k)$, a local belief distribution over a subspace b_i of B is a positive distribution f_i such that

$$\int_{b_i} f_i(x_i) dV_{b_i}(x_i) = 1$$

Representation

- Define a local belief distribution over each probability variable p_{ij} and each utility variable u_{ij}.
- Specify relationships between variables using linear constraints of the form

$$\sum_i a_i x_i \Re b,$$

where $\Re$ is any of the relations $=$, $\leq$, or $\geq$.

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Expected Utility

- Given a decision situation D, the expected utility of an alternative C_i is

$$E(C_i) = \sum_{j=1}^{m_i} p_{ij} u_{ij}$$

- A decision maker adhering to the principle of maximizing expected utility chooses the alternative with the highest expected utility.

Centroids

- ▷ The centroid of a global belief distribution g defined over a unit cube $B = (b_1, \ldots, b_k)$ is the vector $x_g = (\beta_1, \ldots, \beta_k)$ in B whose i:th component is

$$\beta_i = \int_B x_i \cdot g(x) dV_B(x)$$

- ▷ The centroid of a local belief distribution f defined over the interval b_i is

$$x_f = \int_{b_i} x_i \cdot f(x_i) dV_{b_i}(x_i)$$

- ▷ Intuitively, the centroid is where the belief mass is concentrated.

Generalized Expected Mean Value

▷ The generalized expected mean value of an alternative C_i, given centroids $x_{\mathbf{p}_i}$ for the belief distribution over $(p_{i1}, \ldots, p_{im_i})$ and $x_{\mathbf{u}_i}$ for the belief distribution over $(u_{i1}, \ldots, u_{im_i})$, is

$$\begin{aligned} G(C_i) &= \langle x_{\mathbf{p}_i}, x_{\mathbf{u}_i} \rangle \\ &= \sum_{j=1}^{m_i} x_{\mathbf{p}_i(j)} x_{\mathbf{u}_i(j)} \end{aligned}$$

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Inconsistency

Given the vector $x_{\mathbf{p}}$ consisting of the centroids $x_{p_{ij}}$ of the belief distributions over the probability variables, and a set $\mathcal{P}$ of linear constraints, it may be the case that the centroids fall outside $E_{\mathcal{P}}$ (e.g. $\sum_{j=1}^{m_i} x_{p_{ij}} \neq 1$).

When this happens, we can choose a vector $x'_{\mathbf{p}}$ in $E_{\mathcal{P}}$ and use this when computing the generalized mean value.

What is a good choice for $x'_{\mathbf{p}}$?

Measuring Inconsistency

Define an inconsistency measure $M(x)$, and let $x'_{\mathbf{p}}$ be

$$x'_{\mathbf{p}} = \mathsf{argmin}_{x \in E_{\mathcal{P}}} M(x)$$

Preferring Vectors Close to the Centroid

The following inconsistency measure expresses a bias towards vectors in $E_{\mathcal{P}}$ that are close to $x_{\mathbf{p}}$:

$$M(x) = \frac{1}{2}\|x - x_{\mathbf{p}}\|^2.$$

(Constrained Least-Squares Problem)

Rationale: Since $x_{\mathbf{p}}$ represents the center of belief mass, minimizing the Euclidean distance is likely to give the vector in $E_{\mathcal{P}}$ which the decision maker has the highest belief in.

Deficiencies

- Does not handle inconsistent constraints (i.e. when $E_{\mathcal{P}} = \emptyset$).
- Treats all constraints as hard. Only $\sum_{j=1}^{m_i} p_{ij} = 1$ are forced upon the decision maker by the axioms of probability. All other constraints are, just as the belief distributions, expressions of the decision maker's beliefs.

Soft Constraints

- Introduce soft constraints that are allowed to be relaxed by adding positive slack variables ξ_i:
  - $\sum_i a_i x_i \leq b$ becomes $\sum_i a_i x_i \leq b + \xi_i$.
  - $\sum_i a_i x_i \geq b$ becomes $\sum_i a_i x_i \geq b - \xi_i$.
  - $\sum_i a_i x_i = b$ becomes
    $\sum_i a_i x_i \leq b + \xi_i$ and $\sum_i a_i x_i \geq b - \xi_{i+1}$.

Modified Inconsistency Measure

A modified inconsistency measure dealing with soft constraints:

$$\tilde{M}(x, \xi_{\mathbf{p}}) = \frac{1}{2}\|x - x_{\mathbf{p}}\|^2 + C \sum_{i=1}^{\ell_p} \xi_{\mathbf{p}(i)}$$

(Convex Quadratic Programming Problem)

The parameter C allows the decision maker to control the penalty for modifying constraints, a larger C expressing a preference for moving the centroid instead of relaxing constraints.

Thoughts on Inconsistency

- The vector $x'_{\mathbf{p}} - x_{\mathbf{p}}$ (and $\xi_{\mathbf{p}}$ when applicable) can guide the decision maker when trying to reduce the inconsistency in the model.
- The decision maker could be allowed to specify whether any constraints other than $\sum_{j=1}^{m_i} p_{ij} = 1$ should be treated as hard.

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Interval Values

- Assume symmetric belief distributions over intervals.
- Reduces complexity because coordinates of the centroids simply become the midpoints of the intervals.

Point Values

- We only have the constraints $\sum_{j=1}^{m_i} p_{ij} = 1$.
- Interpret point values as being the only points with positive belief.
- Minimizing the Euclidean distance to $x_{\mathbf{p}}$ in accordance with $M(x)$ gives us

$$x'_{\mathbf{p}_i(j)} = x_{\mathbf{p}_i(j)} + \frac{1 - \sum_{j=1}^{m_i} x_{\mathbf{p}_i(j)}}{m_i}$$

Conclusions

- Belief distributions—a versatile representation of impreciseness with simple semantics.
- Inconsistent decision models can be evaluated, and the severeness of the inconsistency is given by an inconsistency measure.
- The inconsistency measure can be changed to express different biases.
- Same principles can be applied to decision models with interval and point values.

Future Research

- ▷ Develop a better intuition for how belief distributions can be used.
- ▷ Learning and updating belief distributions given new evidence.
- ▷ Algorithms for sensitivity analysis.
- ▷ Alternative inconsistency measures (e.g. expressing a bias towards vectors with a high belief).