Acceptance Sampling and its Use in Probabilistic Verification

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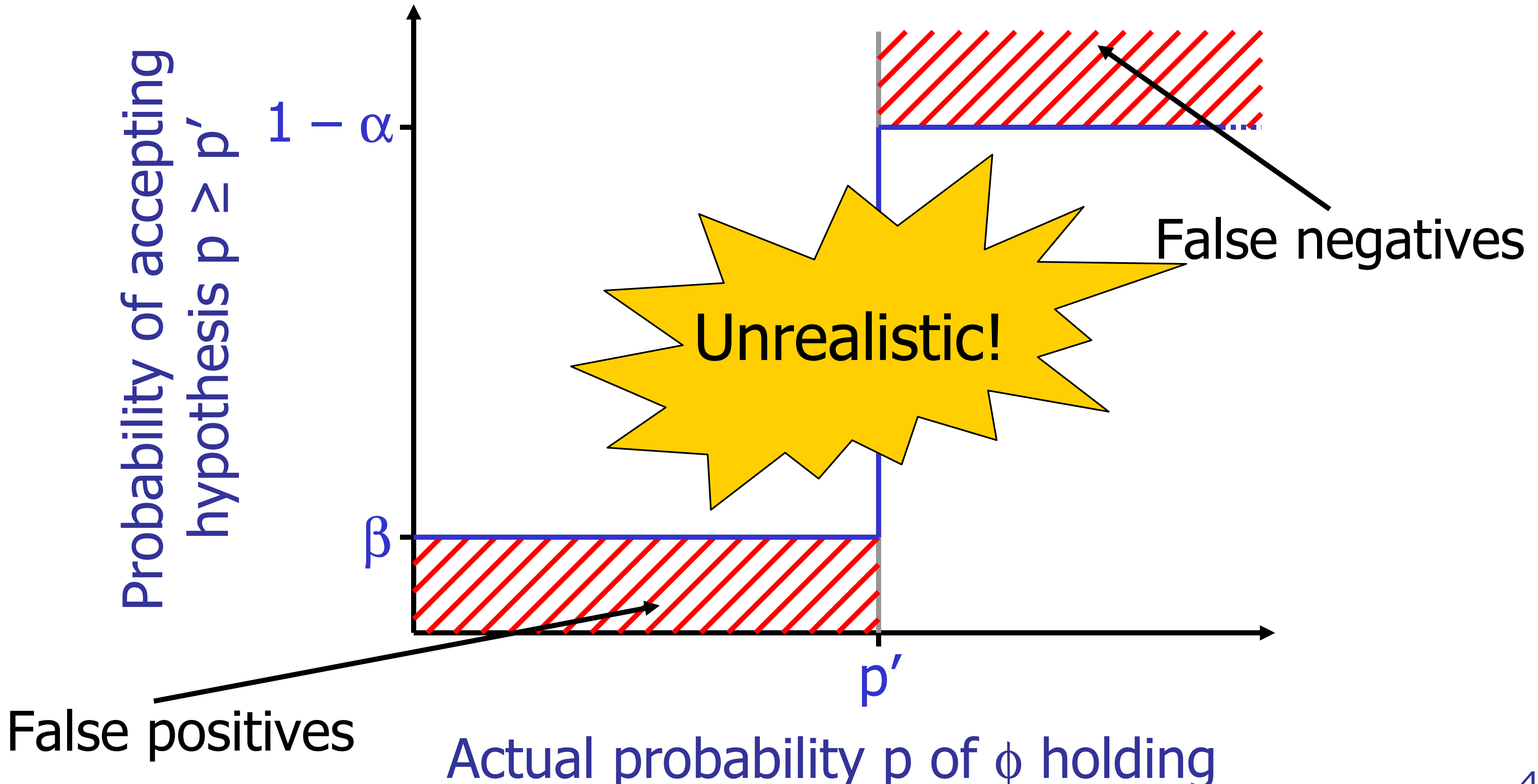
The Problem

- Let ϕ be some property of a system holding with unknown probability p
- We want to approximately verify the hypothesis $p \geq p'$ using sampling
- This problem comes up in PCTL/CSL model checking: $Pr_{\geq p'}(\phi)$
  - A sample is the truth value of ϕ over a sample execution path of the system

Quantifying "Approximately"

- Probability of accepting the hypothesis $p < p'$ when in fact $p \geq p'$ holds: $\leq \alpha$
- Probability of accepting the hypothesis $p \geq p'$ when in fact $p < p'$ holds: $\leq \beta$

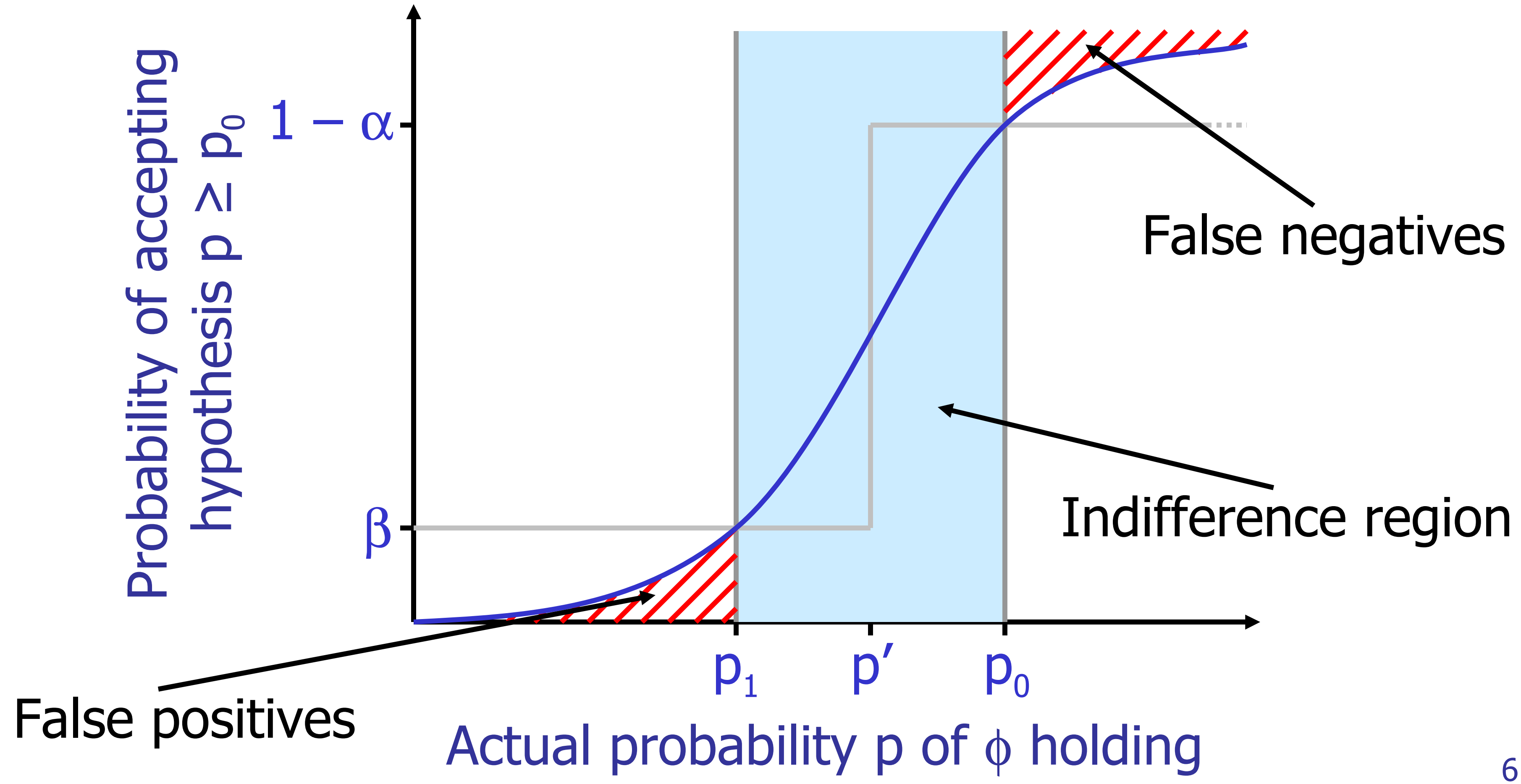
Desired Performance of Test



Relaxing the Problem

- Use two probability thresholds: $p_0 > p_1$
  - (e.g. specify p' and δ and set $p_0 = p' + \delta$ and $p_1 = p' - \delta$)
- Probability of accepting the hypothesis $p \leq p_1$ when in fact $p \geq p_0$ holds: $\leq \alpha$
- Probability of accepting the hypothesis $p \geq p_0$ when in fact $p \leq p_1$ holds: $\leq \beta$

Realistic Performance of Test



Method 1: Fixed Number of Samples

- Let n and c be two non-negative integers such that $c < n$
  - Generate n samples
  - Accept the hypothesis $p \leq p_1$ if at most c of the n samples satisfy ϕ
  - Accept the hypothesis $p \geq p_0$ if more than c of the n samples satisfy ϕ

Method 1: Choosing n and c

- Each sample is a Bernoulli trial with outcome 0 (ϕ is false) or 1 (ϕ is true)
- The sum of n iid Bernoulli variates has a binomial distribution

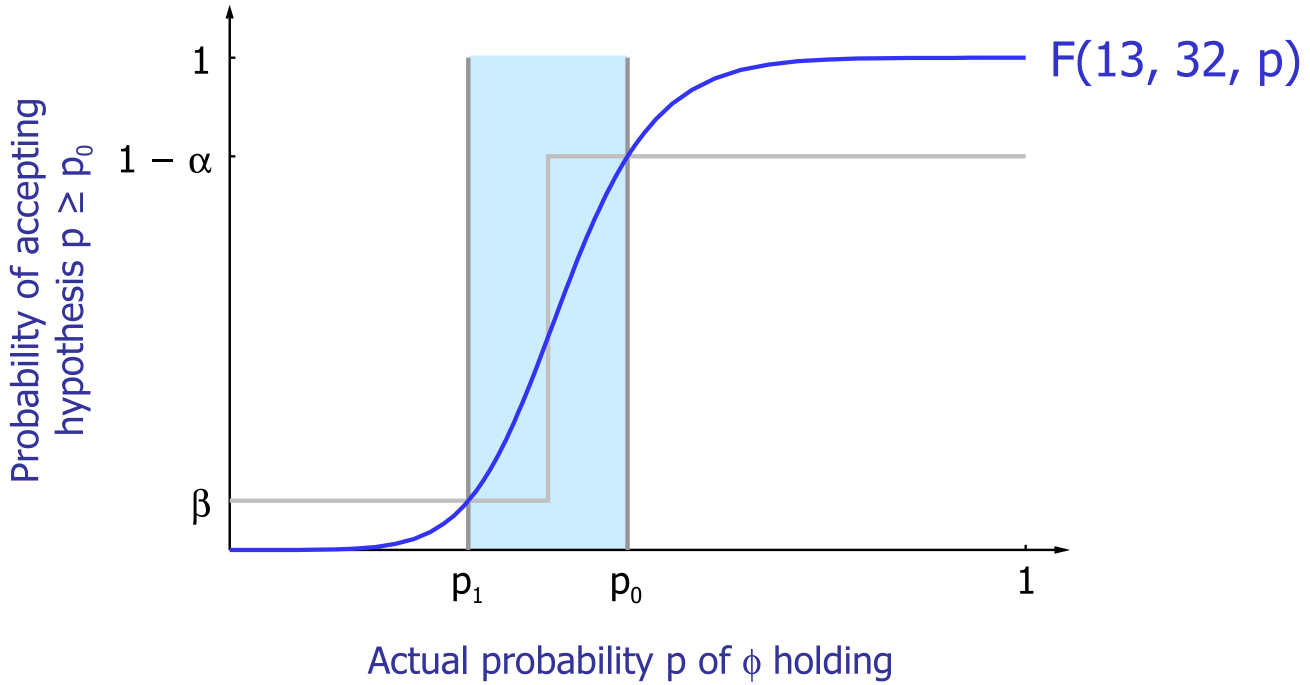
$$F(c,n,p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}$$

Method 1: Choosing n and c (cont.)

- Find n and c simultaneously satisfying:
  1. ~~$\forall p' \in [p_0, 1]$,~~ $F(c, n, p_0) \leq \alpha$
  2. ~~$\forall p' \in [0, p_1]$,~~ $1 - F(c, n, p_1) \leq \beta$
- Non-linear system of inequalities, typically with multiple solutions!
  - Want solution with smallest n
  - Solve non-linear optimization problem using numerical methods

Method 1: Example

- $p_0 = 0.5$, $p_1 = 0.3$, $\alpha = 0.2$, $\beta = 0.1$:
  - Use $n = 32$ and $c = 13$

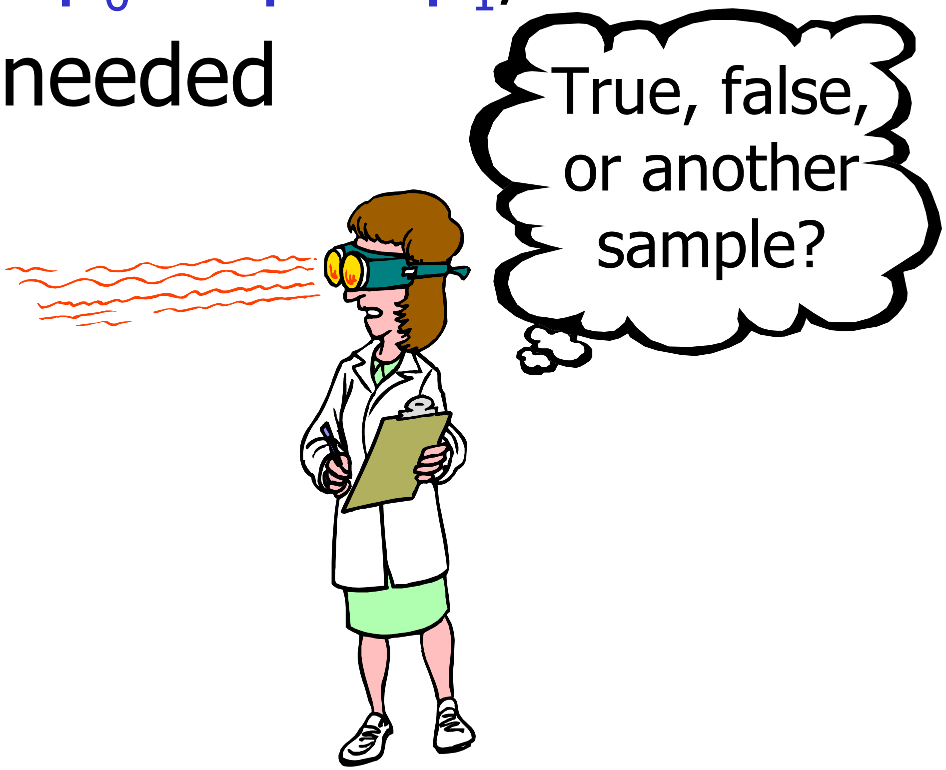


Idea for Improvement

- We can sometimes stop before generating all n samples
  - If after m samples more than c samples satisfy ϕ, then accept $p \geq p_0$
  - If after m samples only k samples satisfy ϕ for $k + (n - m) \leq c$, then accept $p \leq p_1$
- Example of a sequential test
- Can we explore this idea further?

Method 2: Sequential Acceptance Sampling

- Decide after each sample whether to accept $p \geq p_0$ or $p \leq p_1$, or if another sample is needed



The Sequential Probability Ratio Test [Wald 45]

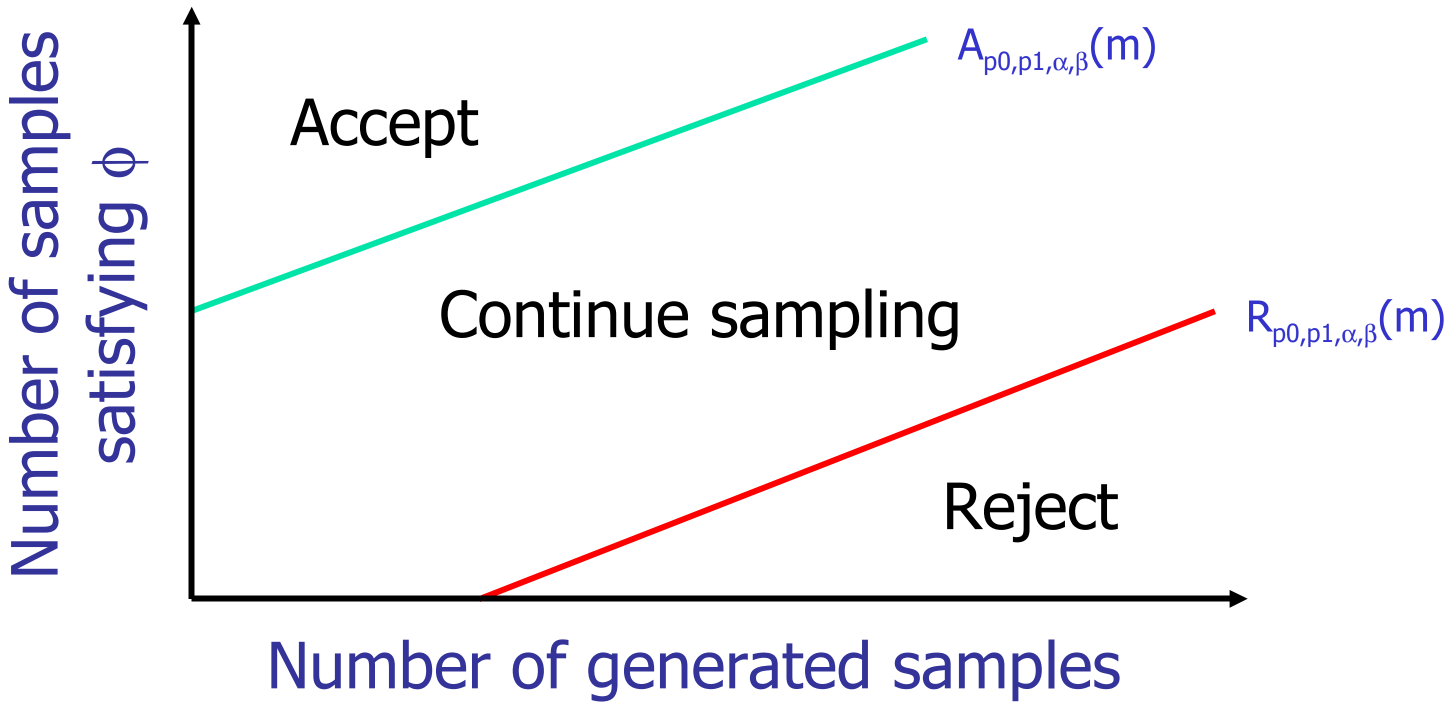
- An efficient sequential test:
  - After m samples, compute the quantity

$$f = \prod_{i=1}^{m} \frac{\Pr[X = x_i \mid p = p_1]}{\Pr[X = x_i \mid p = p_0]}$$

  - Accept $p \geq p_0$ if $f \leq \beta/(1-\alpha)$
  - Accept $p \leq p_1$ if $f \geq (1-\beta)/\alpha$
  - Otherwise, generate another sample

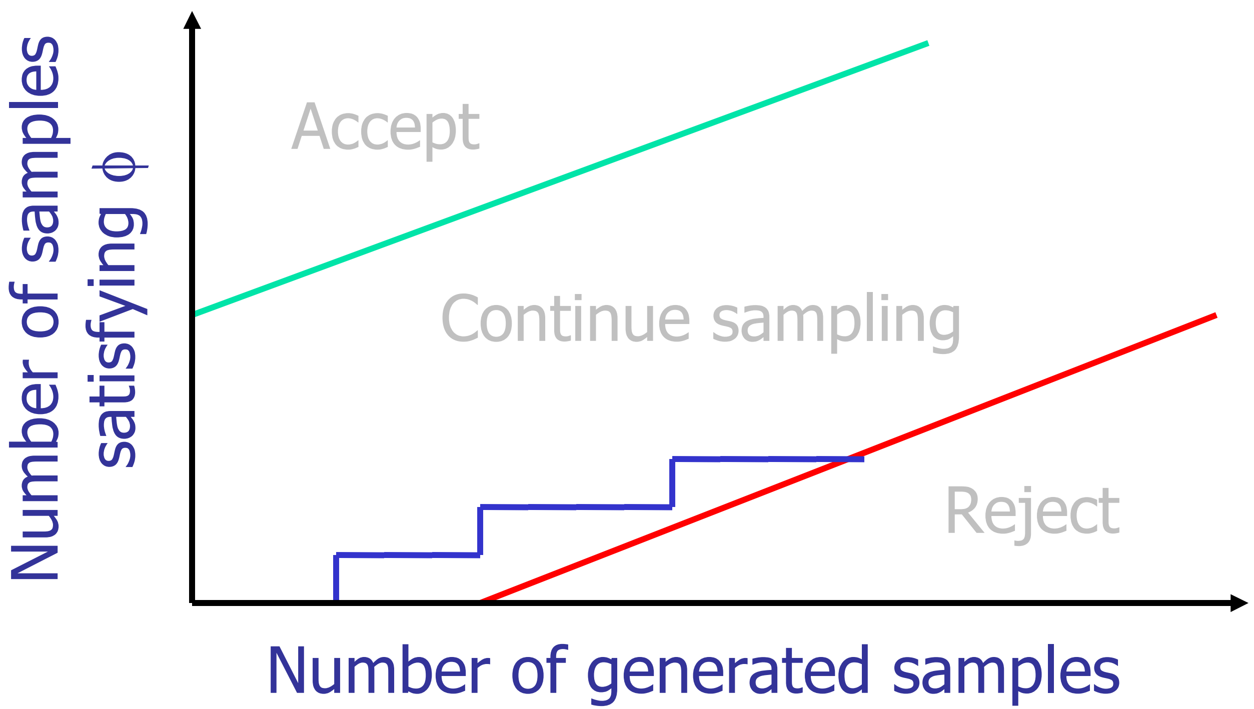
Method 2: Graphical Representation

- We can find an acceptance line and a rejection line give p_0, p_1, α, and β:



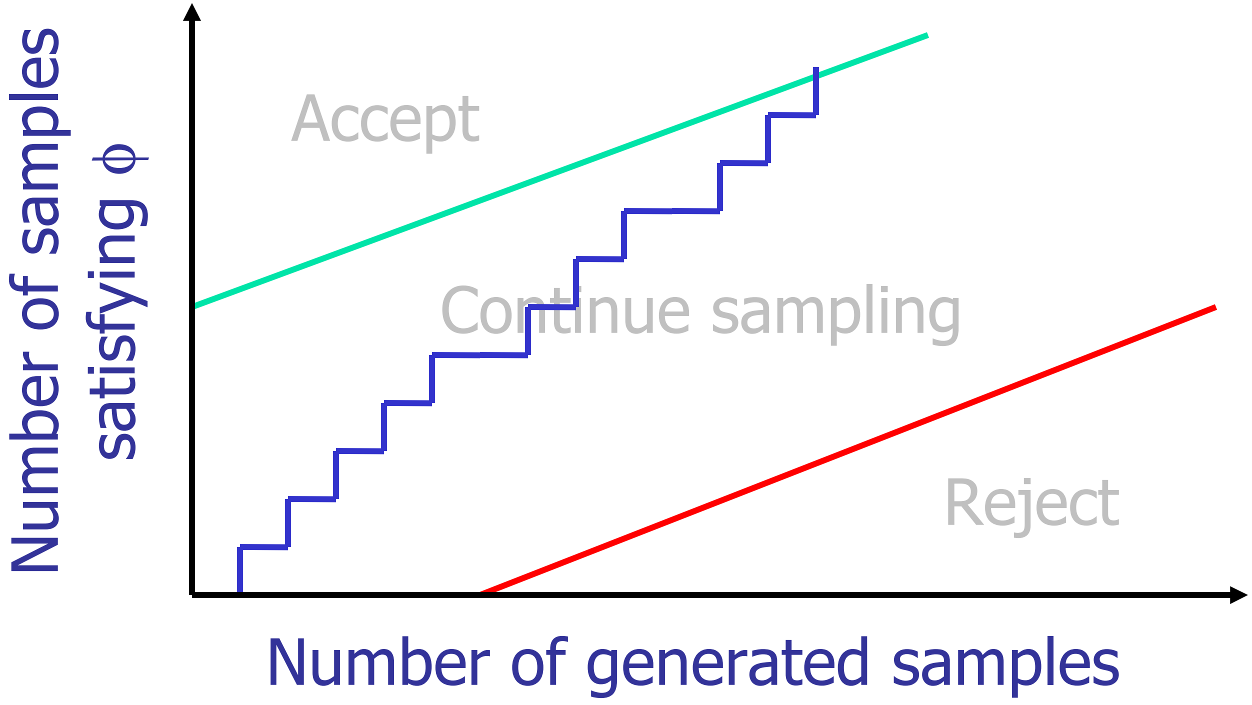
Method 2: Graphical Representation

- Reject hypothesis $p \geq p_0$ (accept $p \leq p_1$)



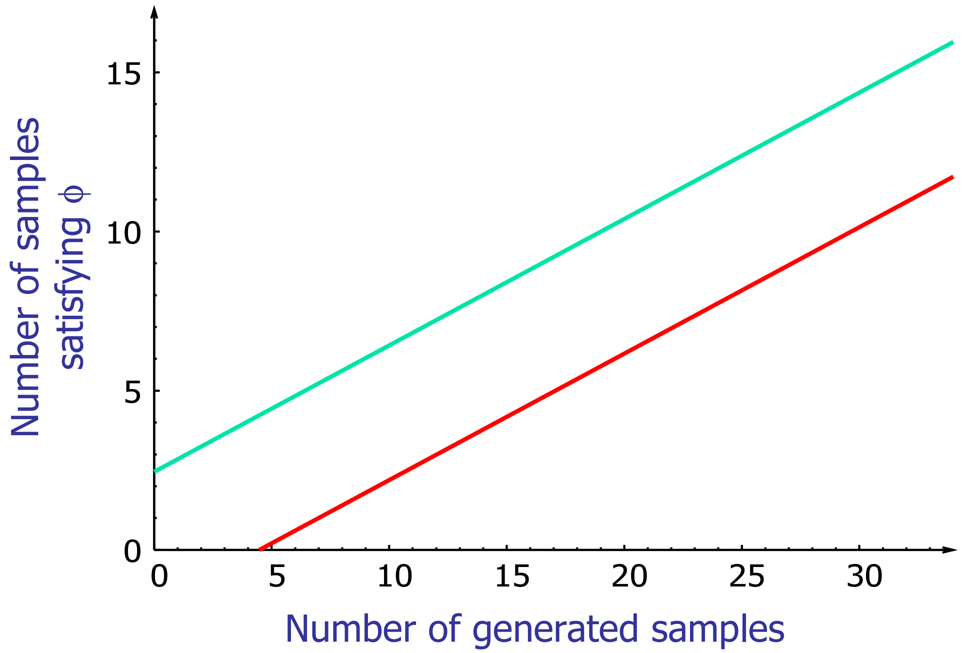
Method 2: Graphical Representation

- Accept hypothesis $p \geq p_0$



Method 2: Example

- $p_0 = 0.5$, $p_1 = 0.3$, $\alpha = 0.2$, $\beta = 0.1$:

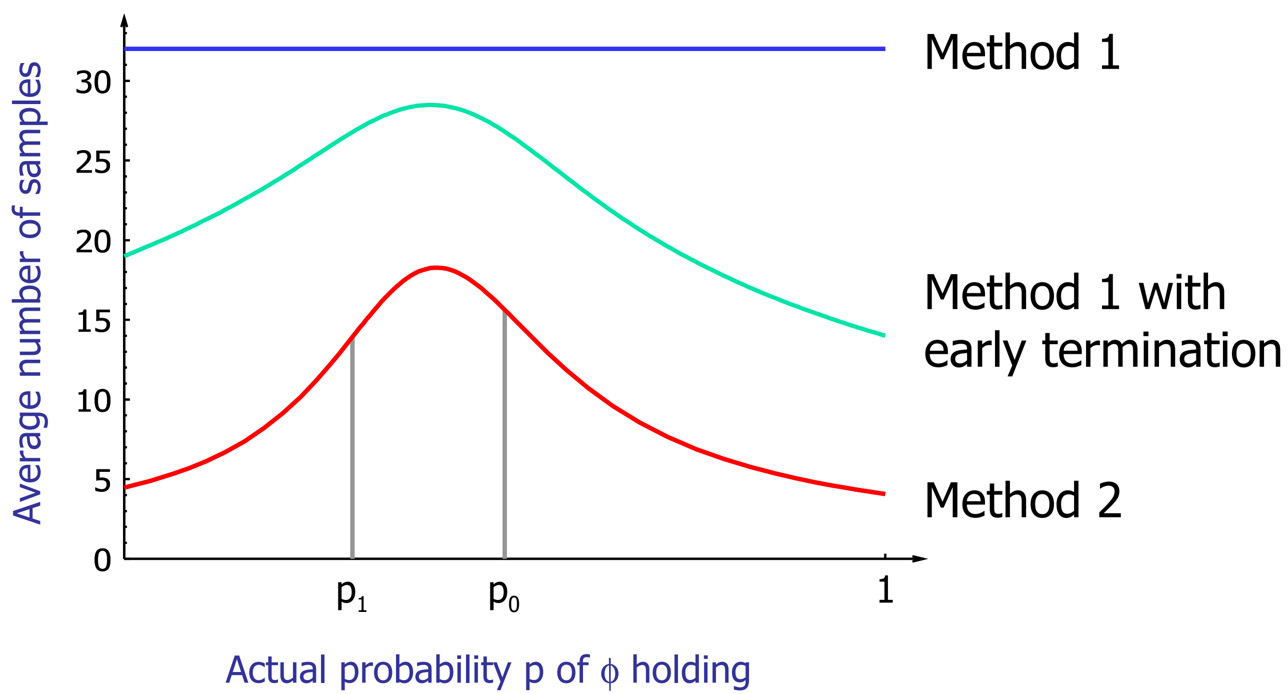


Method 2: Number of Samples

- No upper bound, but terminates with probability one (“almost surely”)
- On average requires many fewer samples than a test with fixed number of samples

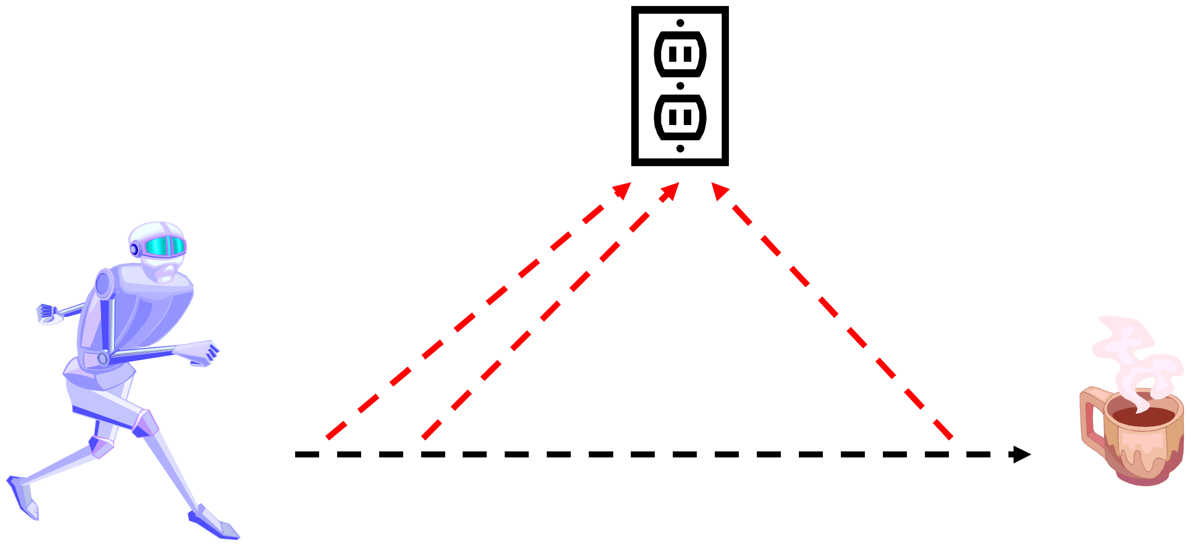
Method 2: Number of Samples (cont.)

- $p_0 = 0.5$, $p_1 = 0.3$, $\alpha = 0.2$, $\beta = 0.1$:



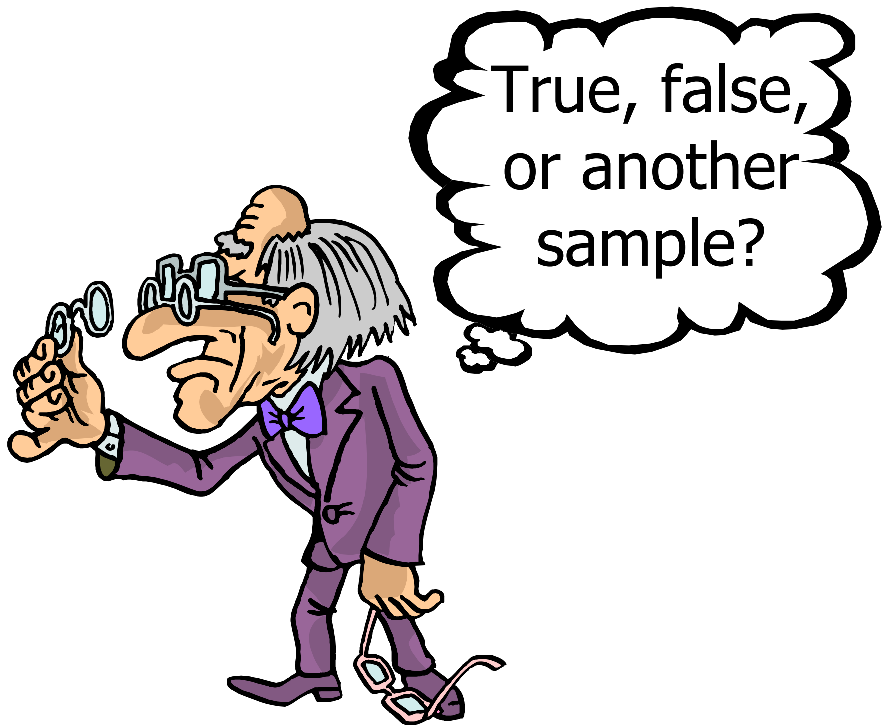
Acceptance Sampling with Partially Observable Samples

- What if we cannot observe the sample values without error?
  - $Pr_{\geq 0.5}(Pr_{\geq 0.7}(\Diamond^{\leq 9}$ recharging$)\ U^{\leq 6}$ have tea$)$



Acceptance Sampling with Partially Observable Samples

- What if we cannot observe the sample values without error?



Modeling Observation Error

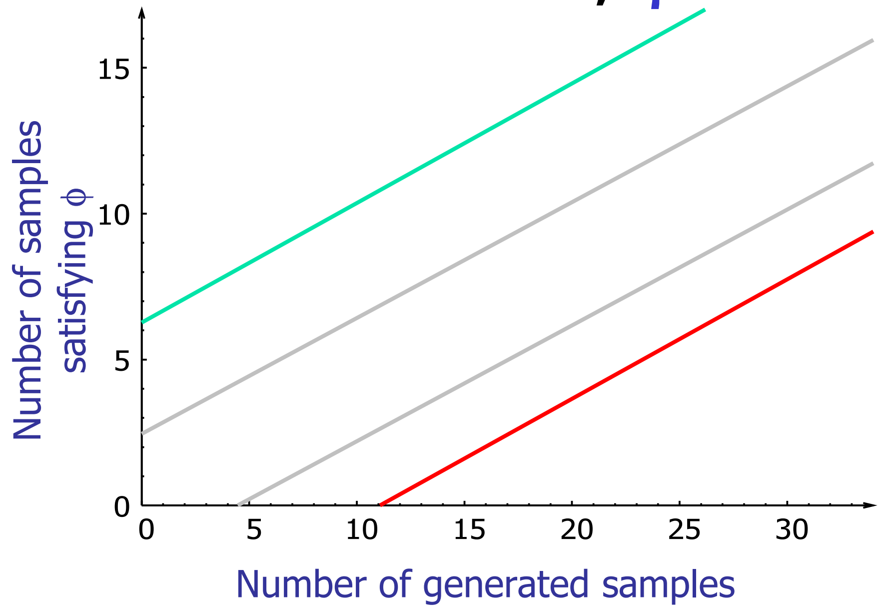
- Assume prob. $\leq \alpha'$ of observing that ϕ does not satisfy a sample when it does
- Assume prob. $\leq \beta'$ of observing that ϕ satisfies a sample when it does not

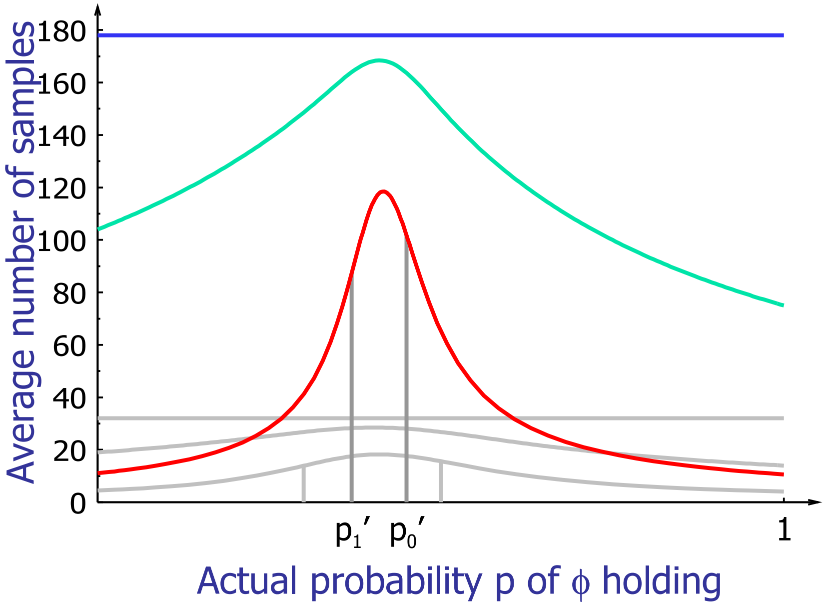
Accounting for Observation Error

- Use narrower indifference region:
  - $p_0' = p_0(1 - \alpha')$
  - $p_1' = 1 - (1 - p_1)(1 - \beta')$
- Works the same for both methods!

Observation Error: Example

- $p_0 = 0.5$, $p_1 = 0.3$, $\alpha = 0.2$, $\beta = 0.1$
- $\alpha' = 0.1$, $\beta' = 0.1$





Application to CSL Model Checking [Younes & Simmons 02]

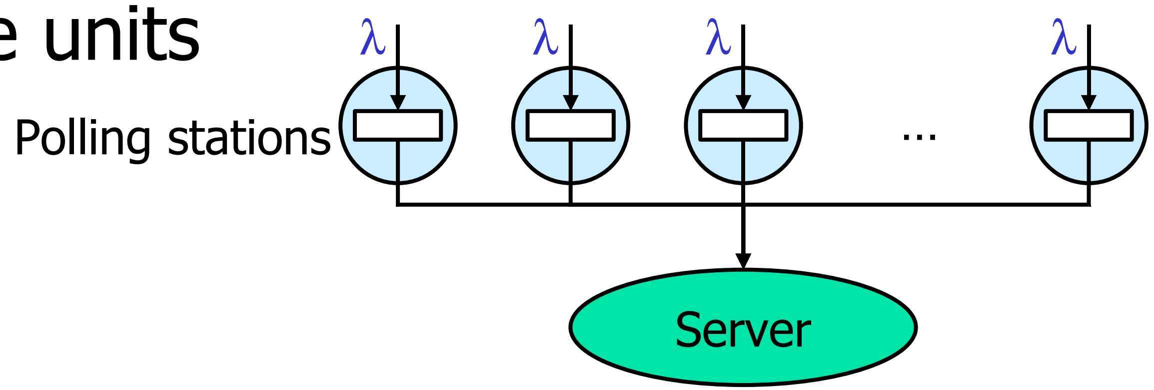
- Use acceptance sampling to verify probabilistic statements in CSL
- Can handle CSL without steady-state and unbounded until
  - Nested probabilistic operators
  - Negation and conjunction of probabilistic statements

Benefits of Sampling

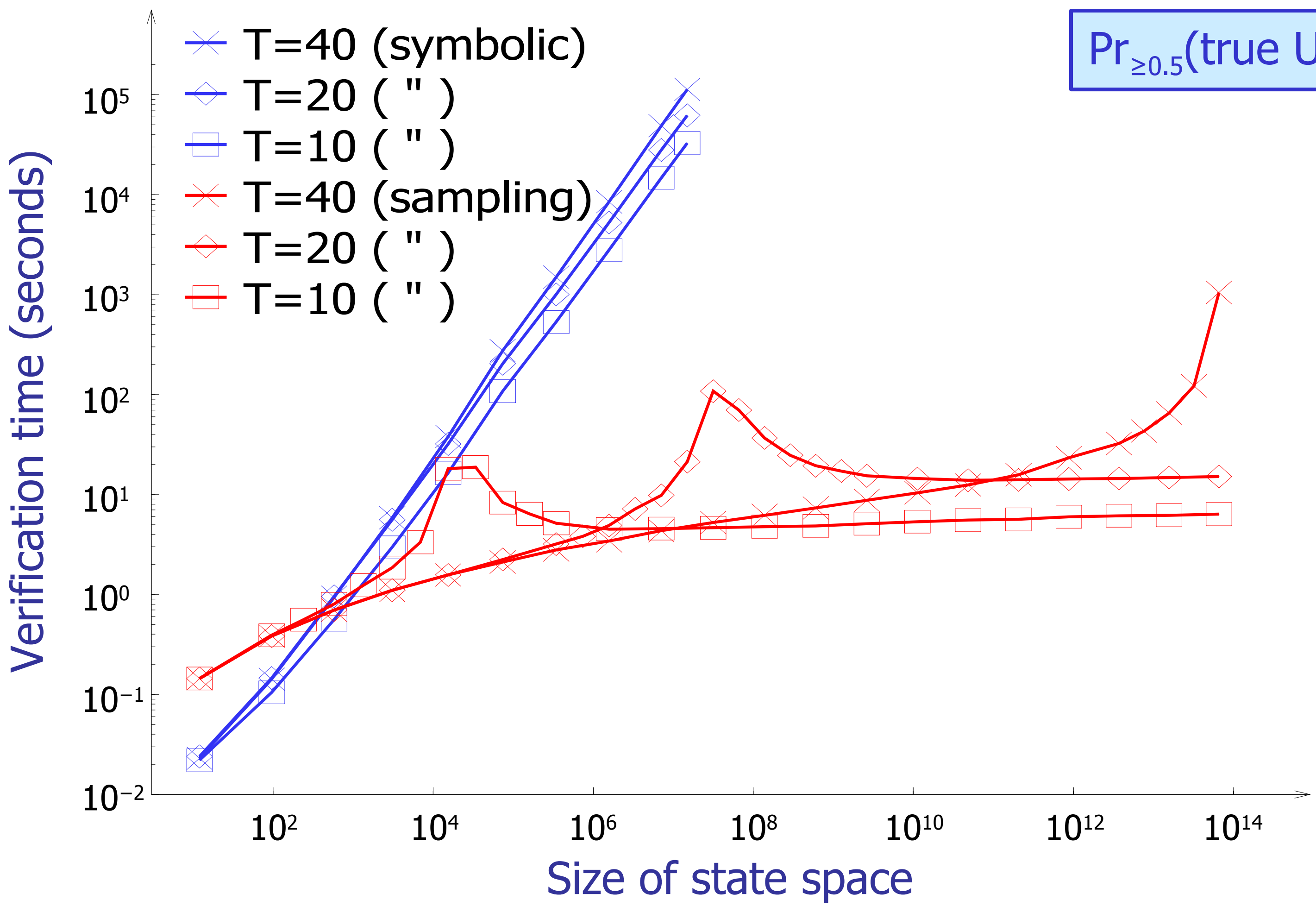
- Low memory requirements
- Model independent
- Easy to parallelize
- Provides “counter examples”
- Has “anytime” properties

CSL Model Checking Example: Symmetric Polling System

- Single server, n polling stations
- State space of size $O(n \cdot 2^n)$
- Property of interest:
  - When full and serving station 1, probability is at least 0.5 that station 1 is polled within t time units

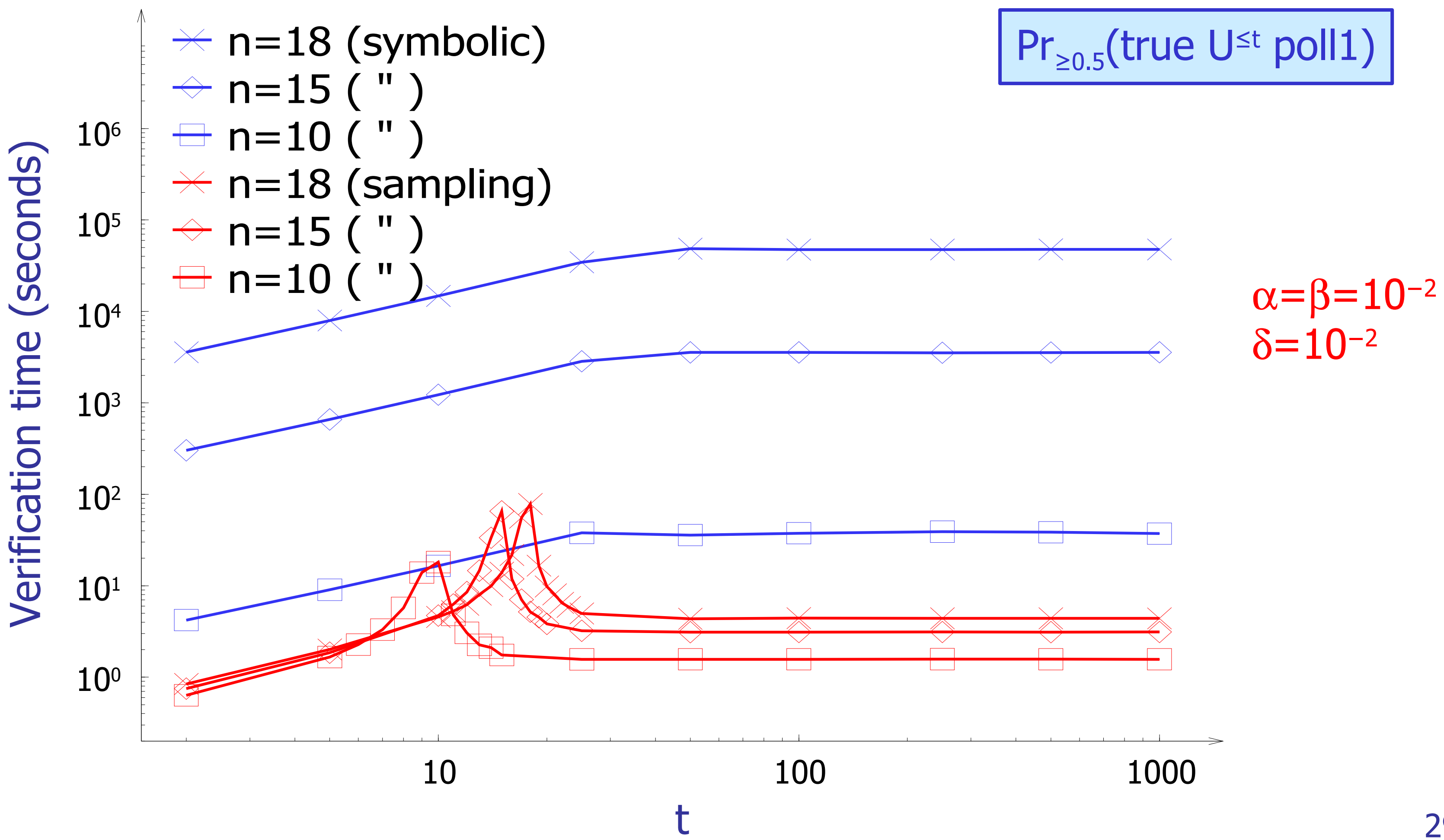


Symmetric Polling System (results) [Younes et al. ??]



$\alpha=\beta=10^{-2}$
$\delta=10^{-2}$

Symmetric Polling System (results) [Younes et al. ??]



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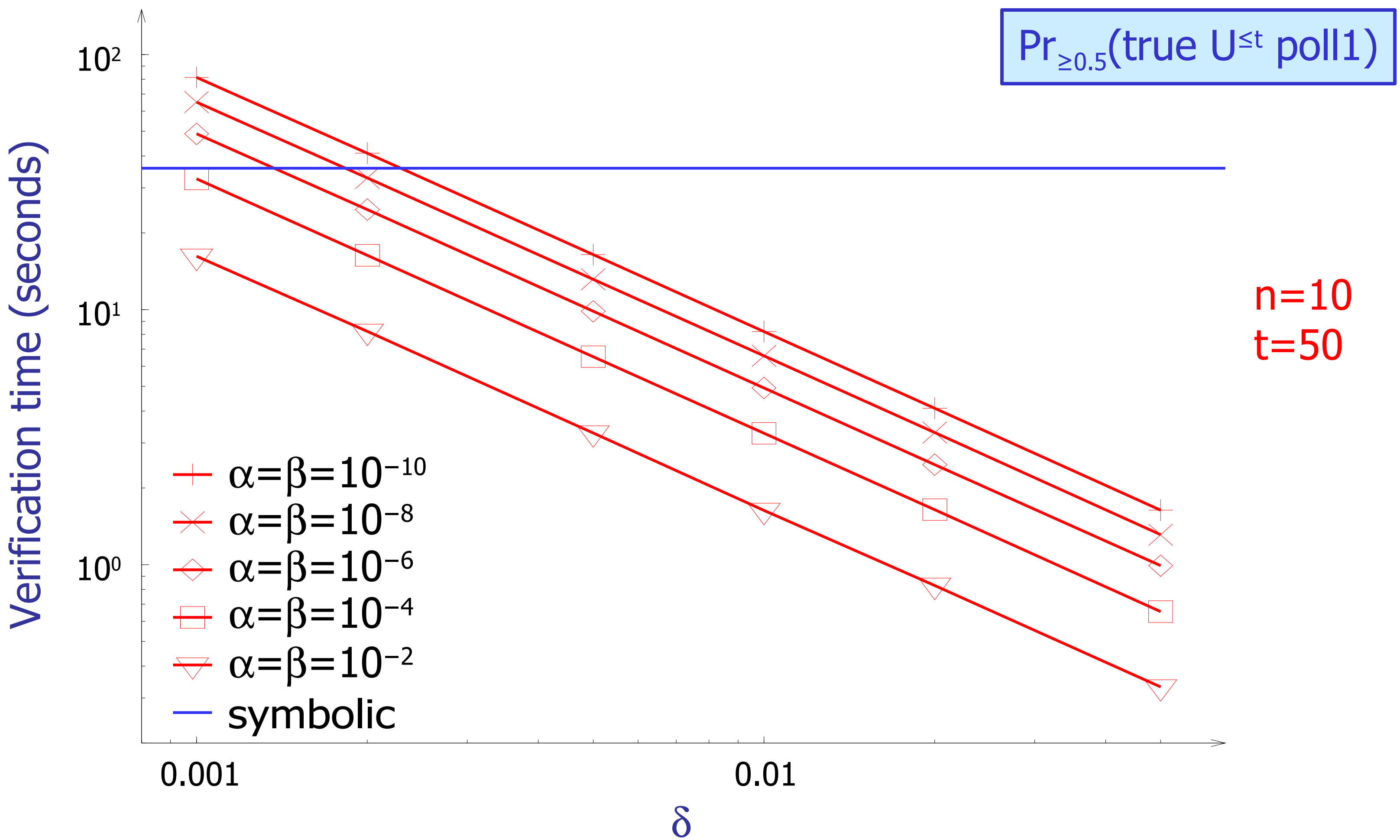

n=10
t=50

Notes Regarding Comparison

- Single state vs. all states
- Hypothesis testing vs. probability calculation/estimation
- Bounds on error probability vs. convergence criterion

Relevance to Planning
[Younes et al. 03]

- Planning for CSL goals in continuous-time stochastic domains
- Verification guided policy search:
  - Start with initial policy
  - Verify if policy satisfies goal in initial state
    - Good: return policy as solution
    - Bad: use sample paths to guide policy improvement and iterate

Summary

- Acceptance sampling can be used to verify probabilistic properties of systems
- Have shown method with fixed number of samples and sequential method
- Sequential method better on average and adapts to the difficulty of a problem